Functional Equations

Functions can be any. Very any and very ugly. They do not have to be linear or polynomial. Functional equations are equations where the unknown is a function.

Main method of solving functional equations is to substitute some values and plug it in equation, even if it would not be used in solution, for example: $x = 0, x = y, x = f(y)$ etc. Sometimes it is beneficial to calculate the function is some set point for example $f(0)$.

Problem 1. (Substitutions) Find every function $f : \mathbb{R} \longrightarrow \mathbb{R}$ that satisfies the following equation:

$$2f(x) + f(1 - x) = 3x$$

Problem 2. (More Substitutions) Find every function $f : \mathbb{R} \longrightarrow \mathbb{R}$ that for any $x, y \in \mathbb{R}$ satisfy the following condition (four different problems)

1. $f(x + y) = f(x) - f(y)$
2. $f(x + y) = f(f(x)) + y$
3. $xf(x) + yf(y) = (x + y)f(x)f(y)$
4. $f(y)f(x) - xy = f(x) + f(y) - 1$

Problem 3. (Functions on Reals) Function $f : \mathbb{R} \in \mathbb{R}$ satisfies the conditions:

1. $f(x + y) = f(x) + f(y)$ for any $x, y \in \mathbb{R}$.
2. $f(1) = 1$.

Find $f(\frac{9}{32})$.

Problem 4. (Limitation) Find every function $f : \mathbb{R} \longrightarrow \mathbb{R}$ that is absolutely limited (which means that there exists such constant C_f that $\forall_{x \in \mathbb{R}} |f(x) \leqslant C_f)$ and that satisfies, for any numbers $x, y \in \mathbb{R}$ the equation: $f(x + y) = f(x) + f(y)$.

Problem 5. (Ugly Domains) Find every function $f : \mathbb{R} \setminus \{0, 1\} \longrightarrow \mathbb{R}$ that for any $x \in \mathbb{R} \setminus \{0, 1\}$ satisfies the following equation:

$$f(x) + f\left(\frac{x-1}{x}\right) = 1 + x$$

Problem 6. (Monotonicity) Find all monotonic functions $f : \mathbb{R} \longrightarrow \mathbb{R}$, that for all $x, y \in \mathbb{R}$ satisfies the equation:

$$f(f(x) - y) + f(x + y) = 0$$

Problem 7. For $x, y, z \in \mathbb{R}$ solve the following system of equations.

$$\begin{cases} x^2 - (y + z + yz)x + (y + z)yz = 0 \\ y^2 - (z + x + zx)y + (z + x)zx = 0 \\ z^2 - (x + y + xy)z + (x + y)xy = 0 \end{cases}$$

Solutions

Here are solutions to (some of) the problems.
Solution 1.

$$2f(x) + f(1-x) = 3x \implies f(x) = \frac{3x - f(1-x)}{2}$$

Now let's substitute $y = 1 - x$

$$f(y) = \frac{3y - f(1-y)}{2} \implies f(1-x) = \frac{3(1-x) - f(x)}{2}$$

Now plugging the result of the equation into the first equation.

$$f(x) = \frac{3x - \frac{3(1-x) - f(x)}{2}}{2}$$

If we calculate this equation we can see that: $f(x) = 3x - 1$.
Solution 2.1
For any two numbers $x, y \in \mathbb{R}$.

$$
\begin{aligned}
& f(x+y) = f(y+x) \\
& f(x) - f(y) = f(y) - f(x) \\
& 2f(x) = 2f(y) \\
& f(x) = f(y) \\
& \implies f \text{ is constant} \\
& \implies f(x+y) = f(x) - f(y) = 0
\end{aligned}
$$

From that we can see that for any number $k \in \mathbb{R}$, $f(k) = 0$.